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Transverse–longitudinal part of the vector potential in classical electrodynamics

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Abstract. Existence of a physical (gauge-invariant) degree of freedom of the vector potential generating no electromagnetic fields is proved in classical electrodynamics within the Dirac generalised Hamiltonian dynamics. The gauge-invariant form of electrodynamics of charged particles is given, within which the question of observing the obtained degree of freedom is discussed. It is shown that it causes an electric current in a superconducting ring put on the solenoid.

1. Introduction

Some 60 years ago Fock (1927) and Weyl (1929) proposed the principle of gauge symmetry for describing the interaction of charged particles with an electromagnetic field. In essence, all modern models of elementary particle physics are based on this principle (Yang and Mills 1954).

The essential element of gauge theories is that the interaction of material fields with a gauge field is accomplished within its potentials defined ambiguously. At first glance, potentials in electrodynamics seem to be unnecessary since equations of motion are written in terms of electromagnetic fields being gauge invariant. (There is another situation in a non-Abelian case where fields are not gauge invariant, and potentials play a more fundamental role (Jackiw 1980). The ambiguity of electromagnetic potentials in quantum mechanics of a charged particle gave rise to an extensive discussion in the literature (Ehrenberg and Siday 1949, Aharonov and Bohm 1959). The essence of the physical problem is to answer the question: what influences a charged particle near the region occupied by the magnetic field (for example, near a solenoid)? If the answer is that a particle interacts with a vector potential, then one may object: a potential is defined with an accuracy of the gradient of an arbitrary function. Hence it cannot play the role of a physical field which influences only a charged particle. There exists an interpretation of this phenomenon in the framework of the non-integrable phase factor (Wu and Yang 1975a, b) (see also Sheikh 1984).

In the present work, we try to look at this problem in the spirit of the quantum theory of gauge fields. It is well known that a gauge gives the first-class constraints (Dirac 1964). In quantum theory the observed are the values commuting with all constraints (Dirac 1964), i.e. these values are gauge invariant. Based on the Dirac formalism used for quantisation of gauge theories, we determine physical degrees of

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freedom in the 'electromagnetic field and charged particles' system and give their physical interpretation (section 2). Then, we show there exists a gauge-invariant field distributed around regions occupied by a magnetic field with which a quantum charged particle interacts as locally as with a vector potential (section 3). In section 4 we suggest the examples of the influence of the gauge-invariant field found on a quantum charged particle, i.e. in essence, we interpret the effects of such as the Aharonov–Bohm effect as a result of the interaction of a charged particle with this field. In particular, the latter induces an electric current in a superconductor.

2. Physical degrees of freedom in electrodynamics

A vector potential A_μ in classical electrodynamics is known to be defined with the accuracy of a gradient of an arbitrary function. Thus, one of four functions A_μ can be removed by a gauge transformation. Nevertheless, an electromagnetic field has only two physical degrees of freedom (two transversal polarisations of a photon). The essence of this paradox has been known for a long time (Dirac 1964, 1967). An electromagnetic field as a dynamical system is a constrained system (Dirac 1964) and, although the gauge arbitrariness contains only one function, it generates two independent constraints for degrees of freedom in electrodynamics. A physical quantity in a gauge theory must not depend on the choice of a gauge (i.e. on an evolution of unphysical variables), so it must be gauge invariant. The latter is equivalent to saying that its Poisson brackets with the first-class constraints are equal to zero (Dirac 1964, 1967).

We use this to give a physical interpretation of the vector potential components in the 'electromagnetic field and charged particles' system in a classical theory as well as in a quantum one. The Lagrangian has the form ($\hbar = c = 1$ here and below)

$$L = \int \mathrm{d}^3 x\, \tfrac{1}{2}(E_n^2 - B_n^2) + \tfrac{1}{2} m\dot{\boldsymbol{r}}^2 + e(\dot{\boldsymbol{r}}, \boldsymbol{A}(\boldsymbol{r})) - eA_0(\boldsymbol{r}) \tag{2.1}$$

where $E_n = -\dot{A}_n - \partial_n A_0$ is an electric field ($n = 1, 2, 3$), B_n are components of a magnetic field $\boldsymbol{B} = \operatorname{rot} \boldsymbol{A}$, $\boldsymbol{r}$ is a position vector of a charged particle, m is its mass, and A_n, A_0 are vector and scalar potentials, respectively. In the case of several particles one should replace $\boldsymbol{r}$, m, e by $\boldsymbol{r}_a$, m_a, e_a and introduce a sum over $\boldsymbol{a}$. We define canonical momenta as follows: $\boldsymbol{p} = \partial L/\partial \dot{\boldsymbol{r}} = \dot{\boldsymbol{r}} + e\boldsymbol{A}(\boldsymbol{r})$ and $\pi^\mu = \partial L/\delta \dot{A}_\mu$ where $\pi^0 = 0$, $\pi_n = -E_n$. So the Hamiltonian is

$$H = \int \mathrm{d}^3 x (\tfrac{1}{2} E_n^2 + \tfrac{1}{2} B_n^2 - A_0 \partial_n E_n) + eA_0(\boldsymbol{r}) + \frac{1}{2m}(p_n - eA_n)^2. \tag{2.2}$$

The Poisson brackets are defined in the standard way:

$$\{C, D\} = \int \mathrm{d}^3 x \left(\frac{\delta C}{\delta A_\mu(x)} \frac{\delta D}{\delta \pi^\mu(x)} - \frac{\delta C}{\delta \pi^\mu(x)} \frac{\delta D}{\delta A_\mu(x)} \right) + \frac{\partial C}{\partial r_n}\frac{\partial D}{\partial p_n} - \frac{\partial C}{\partial p_n}\frac{\partial D}{\partial r_n}. \tag{2.3}$$

Since $\pi_0 = 0$, for the self-consistency of a dynamics one has to require $\dot{\pi}_0 = 0$ at all time moments, i.e.

$$G \equiv \{\pi_0, H\} = \partial_n E_n(\boldsymbol{x}) - e\delta(\boldsymbol{x} - \boldsymbol{r}). \tag{2.4}$$

It is easy to check that $\{G, H\} = 0$; hence (2.4) gives all secondary constraints (Dirac 1964).

It follows from (2.4) that the longitudinal canonical momentum is not a dynamical variable but is determined by a distribution of charges in a system. Therefore, introducing new canonical variables

$$\alpha_n = A_n - \partial_n \partial_k \Delta^{-1} A_k \qquad \xi = \Delta^{-1}\partial_n A_n$$
$$\varepsilon_n = \pi_n - \partial_n \partial_k \Delta^{-1} \pi_k \qquad \pi_\xi = -\partial_n \pi_n \tag{2.5}$$

where $\partial_n \alpha_n = \partial_n \pi_n = 0$ and Δ^{-1} is an operator inverse to the Laplace operator $\Delta = \partial_n \partial_n$ in the whole space $\mathbb{R}^3$, we can formulate the theory in terms of gauge-invariant physical variables α_n and ε_n since $G = \pi_\xi(x) - e\delta(x - r)$, $\{\varepsilon_n, G\} = \{\alpha_n, G\} = 0$. It is well known that the physical degrees of freedom in electrodynamics are transversal components of a vector potential (Dirac 1967). The longitudinal part corresponds to the Coulomb field. Indeed, reform $\int d^3x\, E_n^2 = \int d^3x(\varepsilon_n^2 - \pi_\xi \Delta^{-1}\pi_\xi)$. Now we see that the second term turns into the energy of the Coulomb interaction of charges e_a after a substitution $\pi_\xi = \sum_a e_a \delta(x - r_a)$ and it also contains its own infinite Coulomb energies of the charges (the consequence of the pointlikeness of particles). Note, $\{G, p_n\} \neq 0$; thus, the momentum of a particle is not gauge invariant. However, we can easily remove this trouble by passing to the new canonical momentum $p'_n = p_n + e\partial_n \xi(r)$. One can check that $\{r_n, p'_m\} = \delta_{nm}$ and $\{G, p'_n\} = 0$. Hence, p'_n should be identified with the physical observed momentum of a charged particle.

As a result, we get the physical Hamiltonian of the system after solving all constraints

$$H_{\mathrm{ph}} = \int d^3x\, \tfrac{1}{2}(\varepsilon_n^2 + B_n^2) + \tfrac{1}{2} \sum_{a \neq b} \frac{e_a e_b}{4\pi |r_a - r_b|} + \sum_a \frac{1}{2m_a}[p'_a - e_a \boldsymbol{\alpha}(r_a)]^2 \tag{2.6}$$

where we have omitted the infinite Coulomb energies of charges. One can be convinced that the Hamiltonian equations of motion $\dot{\boldsymbol{\alpha}} = \{\boldsymbol{\alpha}, H_{\mathrm{ph}}\}$, $\dot{\boldsymbol{\varepsilon}} = \{\boldsymbol{\varepsilon}, H_{\mathrm{ph}}\}$ and $\dot{r}_a = \{r_a, H_{\mathrm{ph}}\}$, $\dot{p}'_a = \{p'_a, H_{\mathrm{ph}}\}$ coincide with the standard equations of motion for the 'electromagnetic field and charged particles' system.

Let us turn now to quantum theory to elucidate the role of a longitudinal part of a vector potential in it. This question, of course, was discussed many times in the framework of a field theory (Dirac 1964, 1967, Prokhorov 1988). Nevertheless, we shall try to make our consideration more clear as applied to quantum mechanics of charged particles. The quantisation of the present system is made by changing canonical variables by the operators and $\{,\} \to -i[,]$ ($[,]$ is a commutator). Then

$$[A_\mu(x), \pi^\nu(y)] = i\delta^\nu_\mu \delta(x - y)$$
$$[r^a_n, p^b_k] = i\delta_{nk}\delta_{ab} \tag{2.7}$$

and the operators (2.4) and π_0 pick out a physical (gauge-invariant) Hilbert subspace $\mathcal{H}_{\mathrm{ph}}$ of states (Dirac 1964), i.e.

$$\pi_0|\Phi_{\mathrm{ph}}\rangle = 0 \qquad G|\Phi_{\mathrm{ph}}\rangle = 0. \tag{2.8}$$

To solve (2.8) we use the coordinate representation in which $\pi^\nu(x) = -i\delta/\delta A_\nu(x)$, $p^a_n = -i\,\partial/\partial r^a_n$ and the states Φ_{ph} are functions of r^a and functionals of A_ν. This representation is called the functional representation in quantum field theory (Shweber 1961). The states

$$\Phi_{\mathrm{ph}}[A_\nu, r^a] = F[\alpha, r^a] \exp\left(i \sum_a e_a \Delta^{-1}\partial_n A_n(r^a)\right) \tag{2.9}$$

satisfy (2.8) and present a general solution of them where $\boldsymbol{\alpha}$ is defined in (2.5) and F is an arbitrary functional of $\boldsymbol{\alpha}$ and a function of r^a.

Thereby the longitudinal part of the vector potential determines a phase of a wavefunction of charged particles. We show now that this phase describes the Coulomb field of a quantum charged particle. Let us consider, for example, two quantum charges. The wavefunction

$$\langle \boldsymbol{r}_1, \boldsymbol{r}_2, A_\nu|1, 2\rangle_{\rm ph} = \exp[{\rm i}\, e_1\Delta^{-1}\partial_n A_n(\boldsymbol{r}_1) + {\rm i}\, e_2\Delta^{-1}\partial_n A_n(\boldsymbol{r}_2)]\psi(\boldsymbol{r}_1, \boldsymbol{r}_2) \tag{2.10}$$

corresponds to them in accordance with (2.9) where $|\psi|^2$ defines the density function of charges in their configurational space. Now we calculate the electric-field energy for the state (2.10):

$$\mathscr{E}_{\rm el} = \langle 1, 2|\tfrac{1}{2}\int {\rm d}^3x\, E_n^2|1, 2\rangle \tag{2.11}$$

where $E_n = -\pi_n$ is an operator of an electric field. Using the explicit form of π_n and (2.10) we find

$$\mathscr{E}_{\rm el} = \tfrac{1}{2}\int {\rm d}^3r_1\, {\rm d}^3r_2|\psi(\boldsymbol{r}_1, \boldsymbol{r}_2)|^2 \sum_{a,b=1}^{2} \frac{e_a e_b}{4\pi|\boldsymbol{r}_a - \boldsymbol{r}_b|}. \tag{2.12}$$

Thus, $\mathscr{E}_{\rm el}$ is the Coulomb energy in a system of two quantum charges also including the Coulomb energy of every particle which is infinite ($a = b$ in (2.12)). If particles are localised at the points $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$, i.e. $|1, 2\rangle = |1\rangle|2\rangle$ and $\langle 1|\boldsymbol{r}_1|1\rangle = \boldsymbol{R}_1$, $\langle 2|\boldsymbol{r}_2|2\rangle = \boldsymbol{R}_2$ then

$$\mathscr{E}_{\rm el} = \tfrac{1}{2}\sum_{a,b=1}^{2} \frac{e_a e_b}{4\pi|\boldsymbol{R}_a - \boldsymbol{R}_b|} \tag{2.13}$$

and $\mathscr{E}_{\rm el}$ coincides with the energy of the Coulomb interaction of two classical charges.

Thus, we are convinced that the quantum theory of a longitudinal component of a vector potential describes the Coulomb field of a charged particle. The wavefunctions (2.9) depend on A_n non-locally, which exactly corresponds to the notion about a charged particle with the Coulomb field distributed around it (one can hardly imagine a charged particle without its Coulomb field!). However, we can easily get rid of this non-locality changing the potential in the quantum Hamiltonian. Indeed, since the dependence on ξ (see (2.5)) of physical states is known explicitly, the operator of the longitudinal part of the electric field can be calculated in $\mathscr{H}_{\rm ph}$:

$$\tfrac{1}{2}\int {\rm d}^3x\, E_n^2\Phi_{\rm ph} = \left[-\tfrac{1}{2}\int {\rm d}^3x \frac{\delta^2}{\delta\alpha_n^2(\boldsymbol{x})} + \tfrac{1}{2}\sum_{a,b} \frac{e_a e_b}{4\pi|\boldsymbol{r}_a - \boldsymbol{r}_b|}\right]\Phi_{\rm ph} \tag{2.14}$$

where we have used the operators (2.5). Moreover,

$$(\boldsymbol{p}_a - e_a\boldsymbol{A}(\boldsymbol{r}_a))\Phi_{\rm ph} = (\boldsymbol{p}'_a - e_a\boldsymbol{\alpha}(\boldsymbol{r}_a))\Phi_{\rm ph} \tag{2.15}$$

where $p_n'^a = p_n^a - e_a\partial_n\xi(\boldsymbol{r}_a)$. Equalities (2.14) and (2.15) show that in $\mathscr{H}_{\rm ph}$ we can use the quantum Hamiltonian (2.6). Moreover, as it follows from (2.15) we can omit the dependence of the phase of $\Phi_{\rm ph}$ on ξ assuming that $\boldsymbol{p}'_a = -{\rm i}\,\partial/\partial\boldsymbol{r}_a$ in the Hamiltonian (2.6). Thus we get the usual quantum theory of charged particles where we would like to emphasise charged particles interact locally with the field $\boldsymbol{\alpha}(\boldsymbol{x})$. The remainder of the existence of the longitudinal degree of freedom ξ consists in the appearance of the Coulomb interaction between charged particles.

Thus, only two transversal components α_n of a vector potential A_μ are physical, i.e. gauge invariant. The longitudinal part of A_n is not dynamical and provides the Coulomb interaction of charges. It is the well known result in quantum field theory (Prokhorov 1982, 1988, Faddeev and Jackiw 1988). In conclusion, we should like to make several remarks. The Hamiltonian formalism breaks an explicit Lorentz invariance. However, the initial Lagrangian has it; therefore, it is present implicitly in the Hamiltonian formalism (Schwinger 1962, 1963). Nevertheless, one can suggest an explicit Lorentz-invariant formulation of quantum theory for (2.1) using the quantisation method by Fermi and Dirac with the condition $\partial_\mu A_\mu = 0$ (Dirac 1967) (about the self-consistency of this method see also Prokhorov (1988)). The conclusions are not changed in this case. The main point is that in the operator formulation, the condition $\partial_\mu A_\mu \Phi_{\rm ph} = 0$ should be supplemented by $\partial_0\partial_\mu A_\mu \Phi_{\rm ph} = 0$ (Dirac 1967). So only two degrees of freedom, transversal photons, described by α_n give again a contribution to the physical Hamiltonian of an electromagnetic field (Dirac 1967).

Physical states picked out by the operators of constraints in the Dirac quantisation scheme are unnormalised in some sense. The simplest example is the first equation in (2.8). If $\Phi_{\rm ph}$ does not depend on A_0, the integration of $|\Phi_{\rm ph}|^2$ over A_0 gives an infinite factor. But this problem is not the principal one in electrodynamics (Belinfante 1949, Prokhorov 1988). In the general case, we note that, first, the unnormalisability is due to integration over unphysical variables, which does not influence the physics under any circumstances. Thus, we may ignore this infinite factor defining the scalar product in $\mathscr{H}_{\rm ph}$. Second, we may understand the normalisation of physical states in the full Hilbert space as normalisation of generalised functionals of states (for example, we normalise in this way plane waves, being eigenfunctions of the Hamiltonian for free particles) (Prokhorov 1988, Shabanov 1989).

3. Quasiclassical approach for an electromagnetic field

Now we consider the situation when we may neglect quantum properties of the electromagnetic field in the above system. In this approach

$$F[\boldsymbol{\alpha}, \boldsymbol{r}] = \exp(\mathrm{i}\, S[\boldsymbol{\alpha}])\psi_{\rm ph}(\boldsymbol{r}) \tag{3.1}$$

in (2.9) where, in accordance with the rules of quasiclassical description, $S[\boldsymbol{\alpha}]$ is the action of the electromagnetic field. In this case, the electromagnetic field plays the role of an external field for a quantum particle. Then it follows from (2.6) that the Schrödinger equation for charged particles has the form

$$\left[\sum_a \frac{1}{2m_a}\left(-\mathrm{i}\frac{\partial}{\partial \boldsymbol{r}_a} - e_a\boldsymbol{\alpha}(\boldsymbol{r}_a)\right)^2 + V_{\rm coul}(\boldsymbol{r}_a)\right]\psi_{\rm ph} = E\psi_{\rm ph} \tag{3.2}$$

where $V_{\rm coul}$ is the Coulomb energy interaction of particles with each other and with an external charge, and the electromagnetic energy in (2.6) is included in E. Here $\psi_{\rm ph}$ is a gauge-invariant wavefunction of a system. Needless to say, (3.2) can also be derived from the standard equation:

$$\sum_a \left[\frac{1}{2m_a}\left(-\mathrm{i}\frac{\partial}{\partial \mathbf{r}_a} - e_a\boldsymbol{A}(\boldsymbol{r}_a)\right)^2 + e_aA_0(\boldsymbol{r}_a)\right]\psi = \mathrm{i}\partial_t\psi \tag{3.3}$$

which is invariant under gauge transformations:

$$\psi \to \exp\left(\mathrm{i}\sum_a e_a\omega(\boldsymbol{r}_a)\right)\psi \qquad A_k \to A_k + \partial_k\omega \qquad A_0 \to A_0 - \dot{\omega} \tag{3.4}$$

if one introduces the gauge-invariant wavefunction

$$\psi_{\rm ph}=\exp\Big(-{\rm i}\sum_a e_a\Delta^{-1}\partial_n A_n(\boldsymbol{r}_a)\Big)\psi. \tag{3.5}$$

Indeed, substituting (3.5) in (3.3) and taking into consideration the constraint $\partial_n E_n = -\partial_n \dot{A}_n - \Delta A_0 = \rho$ where ρ is the density of external charges, we get (3.2) if the external fields are stationary, and also

$$V_{\rm coul}=\tfrac{1}{2}\sum_{a\neq b}\frac{e_a e_b}{4\pi|\boldsymbol{r}_a-\boldsymbol{r}_b|}-\sum_a e_a\Delta^{-1}\rho(\boldsymbol{r}_a) \tag{3.6}$$

where the infinite Coulomb energy of charges is omitted.

Equation (3.2) shows that a quantum charged particle interacts locally with the external field $\boldsymbol{\alpha}(\boldsymbol{r})$ being a gauge-invariant physical degree of freedom in electrodynamics, as has been demonstrated above.

4. Free-field potentials in electrodynamics

With the help of the gauge-invariant equation (3.2) we shall consider the situation when a charged particle moves near a region occupied by a magnetic field (Ehrenberg and Siday 1949, Aharonov and Bohm 1959). However, before turning to this question, we shall concentrate our attention on a more detailed determination of the invariant field $\boldsymbol{\alpha}$.

Let external sources not depend on time. So, there exist only static electromagnetic fields in the system. We assume in addition $\rho=0$ and hence $\boldsymbol{E}=0$. Using the static Maxwell equations we find an expression for the vector potential $\boldsymbol{A}$ depending in general on a gauge. Then, the gauge-independent part of it is

$$\alpha_n[\boldsymbol{A}]=A_n(\boldsymbol{x})+\partial_n\partial_k\int_{\mathbb{R}^3}\frac{A_k(\boldsymbol{y})\,{\rm d}^3y}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \tag{4.1}$$

Let the field $B\neq 0$ in a region V. Since the magnetic field is solenoidal the region $V^*=\mathbb{R}^3\setminus V$ is multiconnected. We can write for $\boldsymbol{A}$

$$\boldsymbol{A}(\boldsymbol{x})=\begin{cases}\boldsymbol{A}^V(\boldsymbol{x}) & \boldsymbol{x}\in V \qquad \boldsymbol{B}={\rm rot}\,\boldsymbol{A}^V\\ \boldsymbol{\nabla}\chi(\boldsymbol{x}) & \boldsymbol{x}\in V^* \qquad \boldsymbol{B}=0\end{cases} \tag{4.2}$$

and we assume also that $\boldsymbol{B}$ vanishes smoothly at the boundary ∂V, i.e. $\boldsymbol{A}(\boldsymbol{x})$ is a smooth vector function in $\mathbb{R}^3$. Clearly, this boundary condition relates to a great extent to choosing functions of external sources and is not an additional restriction.

The substitution (4.2) in (4.1) shows that in the case $V^*=\mathbb{R}^3\alpha_n(\boldsymbol{\nabla}\chi)=0$, otherwise $\alpha_n=\partial_n\chi^{\rm ph}(\boldsymbol{x})$ at $\boldsymbol{x}=V^*$ where $V^*\subset\mathbb{R}^3$,

$$\chi^{\rm ph}(\boldsymbol{x})=\partial_n\int_V\frac{{\rm A}_n^V(\boldsymbol{y})\,{\rm d}^3y}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}-\partial_n\oint_{\partial V}\frac{\nu_n(\boldsymbol{y})\chi(\boldsymbol{y})}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\,{\rm d}\sigma_y \tag{4.3}$$

where ν_n is the external normal to the surface ∂V and ${\rm d}\sigma_y$ is an element of ∂V. Therefore, the gauge-invariant field α_n is not equal to zero in V^* (i.e. where $\boldsymbol{B}=0$). Also α_n is the gradient of a harmonic function $0\equiv\partial_n\alpha_n=\Delta\chi^{\rm ph}$.

We may connect the invariant $\chi^{\rm ph}$ with the so-called non-integrable phase factor by Wu and Yang (1975a, b) which contains all gauge-invariant information in gauge theories (Jackiw 1980) if we note that a magnetic flux

$$\Phi = \oint_C (\boldsymbol{A}, \mathrm{d}\boldsymbol{l}) = \oint_C (\boldsymbol{\alpha}, \mathrm{d}\boldsymbol{l}) \tag{4.4}$$

is gauge invariant (the phase of the Wu-Yang factor). (The gauge function ω in (3.4) should be 1-valued in order for $\oint (\nabla\omega, \mathrm{d}\boldsymbol{l}) = 0$.) Since V^* is a multiconnected region, a magnetic flux through a contour C uncontracted into a point in V^* is not equal to zero; hence

$$\chi^{\mathrm{ph}}(\boldsymbol{x}) = f(\boldsymbol{x})\Phi \tag{4.5}$$

where f is a harmonic function in V^* determined from the solution of the external Neumann problem ($\nabla\chi^{\mathrm{ph}}$ is defined at the boundary ∂V by the condition of smoothness (see the text after (4.2)).

Obviously, χ^{ph} is a multivalued function in V^* since the circulation of its gradient does not vanish. Note that $\alpha_n = \partial_n\chi^{\mathrm{ph}} \neq 0$ just because of this. One cannot eliminate from $\boldsymbol{A}$ the gradient of a multivalued function by a gauge transformation with a 1-valued ω in (3.4). So this transversal-longitudinal part of the vector potential is its physical degree of freedom. Also, we conclude that there exists a gauge-invariant field $\alpha_n = \partial_n\chi^{\mathrm{ph}}$ distributed near regions occupied by a magnetic field. A magnetic flux is its source in the same way charges and currents are sources for electric and magnetic fields, respectively.

It is easy to check that a classical particle does not feel χ^{ph}. Indeed, substituting $\alpha_n = \partial_n\chi^{\mathrm{ph}}$ in (2.6) (the electromagnetic field Hamiltonian should be omitted since an a electromagnetic field is assumed external, i.e. non-dynamical) and finding the Hamiltonian equations of motion, we see that they coincide with the equations for a free particle. The main point here is that in classical mechanics an influence has a power character but a force acting on a charged particle in the external field is the Lorentz force depending only on strengths $\boldsymbol{E}$ and $\boldsymbol{B}$ being zero in this case.

Nevertheless, χ^{ph} has a transparent physical sense in the classical theory. The Coulomb field of a charged particle penetrates into the region V; therefore, the electromagnetic momentum in V differs from zero (Konopinsky 1978)

$$\boldsymbol{P}(\boldsymbol{r}) = \frac{1}{4\pi}\int_V \mathrm{d}^3x \left[\nabla \frac{e}{|\boldsymbol{r}-\boldsymbol{x}|} \times \boldsymbol{B}\right] \tag{4.6}$$

where $\boldsymbol{r} \in V^*$. After some simple transformations we find

$$\boldsymbol{P}(\boldsymbol{r}) = e\nabla\chi^{\mathrm{ph}}(\boldsymbol{r}). \tag{4.7}$$

For this, one has to integrate (4.6) by parts and then to use the stationary Maxwell equation rot $\boldsymbol{B} = -\Delta\boldsymbol{\alpha} = -\boldsymbol{J}^{\perp}$ where $\boldsymbol{J}^{\perp}$ is a transversal part of an external current giving rise to a magnetic field into V and $\boldsymbol{\alpha} = \nabla\chi^{\mathrm{ph}}$. Thus in the classical theory χ^{ph} gives the moment of the electromagnetic field in 'the charge and a stationary solenoid system'.

5. Quantum theory and superconductivity

Let us consider a quantum particle moving near a region V occupied by a magnetic field. The Schrödinger equation has the form (3.2) where one has to omit V_{coul} (one particle and $\rho = 0$) and to put $\alpha_n = \partial_n\chi^{\mathrm{ph}}$. If it is assumed that a particle wavefunction is 1-valued, then χ^{ph} can change the spectrum of a system (otherwise the substitution

$\psi_{ph} = \exp(ie\chi^{ph})\varphi$ reduces (3.2) to the equation for a free particle). Indeed, we take for example an infinite solenoid directed along the Oz axis. Then, $f(x) = \theta/2\pi$ in (4.4) where θ is the angle of the cylindrical system of coordinate $x \to (r, z, \theta)$. Further we write (3.2) for the quantum rotator (Peshkin *et al* 1961, Peshkin 1981)

$$\frac{1}{2I}\left(L_z - \frac{e\Phi}{2\pi}\right)^2 \psi_{ph} = E_e\psi_{ph} \tag{5.1}$$

where I is a moment of inertia and $L_z = -\mathrm{i}\partial_\theta$ is the operator of the angular momentum projection on the axis Oz. If now $\psi_{ph}(\theta + 2\pi) = \psi_{ph}(\theta)$, i.e. it is 1-valued, $\psi_{ph}(\theta) \sim \exp(\mathrm{i}l\theta)$, l is an integer and $E_l = (1/2I)(l - e\Phi/2\pi)^2$. So the rotator spectrum E_l depends on χ^{ph} at non-integer $l - e\Phi/2\pi$. In our opinion superconductivity provides a more simple and obvious example of these phenomena.

The free energy of a superconductor of volume V_s in the external electromagnetic field is given by the Ginzburg–Landau functional:

$$F_s = \int_{V_s} \mathrm{d}^3x \left\{ -a|\psi|^2 + \tfrac{1}{2}b|\psi|^4 + \frac{1}{4m}\psi^*(-\mathrm{i}\nabla - 2eA)^2\psi \right\} \tag{5.2}$$

where ψ is the complex order parameter or the wavefunction of the Cooper pair in the BCS model, m is the electron mass and e is its charge. Our further analysis will be qualitative. We shall not solve the Ginzburg–Landau equation exactly. However, for our purposes it will be quite enough (note also that we neglected the depth of penetration of the field $\boldsymbol{B}$ into the superconductor). A matter of principle for us is the following: can the field χ^{ph} have a possibility to influence a real physical system or not? In other words, is there a method of elucidating whether the magnetic field exists inside the solenoid without any manipulations with the solenoid itself?

When external fields are stationary, the state of the superconductor is specified by the minimum of its free energy. If the fields are absent, then the absolute minimum is reached at $\psi = \psi_0 = \text{constant}$, when the kinetic energy assumes zero values, and $|\psi_0| = a/b$. Usually, ψ_0 is normalised as $|\psi_0| = n_s/2$ where n_s is a number of paired electrons per unit volume (in general, $|\psi|^2$ is the density of Cooper pairs).

Consider a ring made of a superconducting material and put on the solenoid. If the ring temperature is $T > T_c$, where T_c is the critical temperature, the current in the ring dies down. Let us stabilise the magnetic flux for $T > T_c$. Then, upon cooling the ring to $T < T_c$ it transforms into a superconducting state. Now we shall find the wavefunction of the ground state of a superconductor in this case.

The free energy F_s is invariant under gauge transformations (3.4) (where $e \to 2e$). Using substitution (3.5) ($a = 1$, $e \to 2e$), we rewrite F_s within gauge-invariant quantities α and ψ_{ph} (the function ψ_{ph} is interpreted as above, i.e. it describes the Cooper pair with its Coulomb field), and then we pass to cylindrical coordinates in the kinetic energy operator. Since $\boldsymbol{\alpha} = \nabla\chi^{ph} = \boldsymbol{e}_\theta\Phi/2\pi r$, $\boldsymbol{e}_\theta$ is a basis vector of the cylindrical coordinate system, we may assume that the minimum is reached on the function ψ_{ph} independent both of z and r, i.e. instead of the kinetic term in (5.2) we can write $(4mr^2)^{-1}\psi^*_{ph}(-\mathrm{i}\partial_\theta - \Phi/\Phi_0)^2\psi_{ph}$ where $\Phi = \pi/e$ is the magnetic flux quantum (fluxon). So the only difference of F_s, as compared with the case $\chi^{ph} = 0$, consists in the charge of the rotation energy of the condensate, but it is quite similar to (5.1). Therefore, we conclude that

$$\psi_{ph} = \psi_0 \exp(\mathrm{i}l_0\theta) \qquad l_0 = [\Phi/\Phi_0] \tag{5.3}$$

gives the minimum of F_s (if, certainly, ψ_{ph} is assumed 1-valued). Here, l_0 is the integer to be chosen from the condition of minimal F_s; $[\Phi/\Phi_0]$ means rounding Φ/Φ_0 to the nearest integer.

Let us now calculate the density of a superconducting current:

$$J_s = \frac{e}{2m}[\psi^*_{ph}(-i\nabla - 2e\alpha)\psi_{ph} + hc] \tag{5.4}$$

for the state (5.3). We find

$$J_s = e_\theta \frac{en_s}{2mr_0}(l_0 - \Phi/\Phi_0) \tag{5.5}$$

where r_0 is the ring radius. Note that for a multivalued $\psi_{ph} = \exp[i(\Phi/\Phi)\theta]$, $J_s = 0$. It follows from (5.5) that J_s can have different directions independent of l_0. If $\Phi = N\Phi_0 + \frac{1}{2}\Phi_0$, where N is an integer, the system turns out to be in the state with unsteady equilibrium as F_s is identical for both the cases $l_0 = N$ and $l_0 = N+1$. After 'throwing down' the system into the minimum the current J_s becomes maximum in an absolute value.

The appearance of the current in the situation described above can be explained in a sufficiently simple way. The magnetic flux passing through the superconducting ring should be quantised, i.e. it should be divisible by Φ_0. Thereby, after a passage of the ring into a superconducting state, the current arises in it, the magnetic field of which supplements the total magnetic flux through the ring to an integer number of quanta. Although (5.5) may only serve as the first approach to the solution of the Ginzburg-Landau equation, nevertheless, it gives a correct qualitative picture of the phenomenon. But for quantitative estimates of the current I_s the known formula $I_s = (\Phi_0 l_0 - \Phi)/L$, where L is the contour inductance, can always be used. This formula is exact for superconductors (Fock 1932). We would like to add the following question to the reasonings used: which real physical field makes the Cooper pairs move, thus creating a current? We have seen that the existence of the gauge-invariant physical field χ^{ph} permits us to give the explanation of this phenomenon in the framework of the field theoretical interpretation of interacting matter. The current is just the direct consequence of the local interaction of the field χ^{ph} with charged Cooper pairs. From this point of view, we can also interpret the Aharonov-Bohm effect as the result of the scattering of a charged particle by the field χ^{ph}.

In conclusion we note that the behaviour of a superconductor in the field of a solenoid was considered by Liang and Ding (1988) (see also the discussion about their paper (Tonomura and Fukuhora 1989, Liang and Ding 1989)).

6. Conclusion

As a matter of fact, the field χ^{ph} can produce a mechanical influence. Consider two simple examples. Let the frame made of a suitable material be hung in a vertical plane and the toroidal solenoid be run through the frame (the toroidal solenoid can be taken to exclude the return magentic flux). Moreover, let the external, constant, homogeneous magnetic field $\boldsymbol{B}_0$ be run through the frame. Put the frame so that the flux of $\boldsymbol{B}_0$ through it would be quantised. Then, after cooling this frame to $T < T_c$, it begins to oscillate if the flux inside the solenoid is not quantised. The frequency of small oscillation $\omega = (I_s\Phi_{ext}/I)^{1/2}$ is the superconducting current in the frame, Φ_{ext} is the flux of $\boldsymbol{B}_0$

through it and I is its inertial moment with respect to the hanging axis (oscillation of the frame with a current in the external magnetic field). Also, the turning scales can be used for observing mechanical recoil in the superconducting ring when a superconducting current appears after cooling. When the Cooper pairs begin to move coherently, the atomic frame of the ring gets a recoil in accordance with the conservation of angular momentum. So oscillations of the turning scales will arise, the amplitude of which will be $2\pi I_s m_e/\omega eM$; M, ω, m_e are the ring mass, the natural frequency of the turning scales, and the electron mass, respectively.

The literature devoted to the Aharonov-Bohm effect interpretation, as we have noted above, is very large. One has to recognise that the main direction there is the aspiration to give a gauge-invariant formulation of this phenomenon. However, it is usually done by using the 'string' formulation of electrodynamics (by path-dependent integral) (De Witt 1962, Mandelstam 1962). The direct application of this technique to the Aharonov-Bohm effect was given, for example, in Sheikh (1984), Kazes *et al* (1983), and Lee *et al* (1983). Also, Lee and his co-authors even suggested modifying the Lagrangian of interaction of a charged particle with an electromagnetic field by the change $\boldsymbol{A}\rightarrow\boldsymbol{A}'=\boldsymbol{A}-\nabla\int_C(\boldsymbol{A},\mathrm{d}\boldsymbol{l})$ with the purpose of making it explicitly gauge invariant (but at the cost of its locality). Note, however, that although $\boldsymbol{A}'$ is gauge-invariant, its value depends completely on the contour C being, in general, arbitrary. So it has no clear physical sense.

Our interpretation is based on the gauge-invariant formulation of electrodynamics as the field theory (Prokhorov 1982, 1988, Faddeev and Jackiw 1988) when all gauge-invariant physical degrees of freedom are determined from the analysis of constraints. This approach, as has been shown above, permits one to give to all gauge-invariant variables, used in the theory, a transparent physical sense that, in our opinion, is the main benefit of our formulation.

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